The Klein–Gordon equation of generalized Hulthén potential in complex quantum mechanics

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Abstract
We have investigated the reality of exact bound states of complex and/or
PT-symmetric non-Hermitian exponential-type generalized Hulthén potential.
The Klein–Gordon equation has been solved by using the Nikiforov–Uvarov
method which is based on solving the second-order linear differential equations
by reduction to a generalized equation of hypergeometric type. In many cases
of interest, negative and positive energy states have been discussed for different
types of complex potentials.

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1. Introduction

In the last few years there has been considerable work on PT-symmetric quantum mechanics.
The PT-symmetric formulation manifests itself in real effects of non-Hermitian theories and
also puts some requirements to the Hamiltonian systems. Additionally, insights may frequently
be gained using PT-symmetry, which lead to a deeper understanding of the complex systems.
Following the early studies of Bender and his co-workers [1], the PT-symmetric formulation
has been successfully utilized by many authors [2–10]. The PT-symmetric but non-Hermitian
Hamiltonians have real spectra whether the Hamiltonians are Hermitian or not. Non-Hermitian
Hamiltonians with real or complex spectra have also been analysed by using different methods
[3–6,10–12]. Non-Hermitian but PT-symmetric models have applications in different research
areas, such as nuclear physics [13], condensed matter [14] and population biology [15]. More
recently, to overcome the weaker conditions of PT-symmetry Bender *et al* [16] emphasized
new conditions in the context of complex quantum mechanics. Furthermore, in their study,
including the standard condition of hermiticity, the converse of the fundamental CPT-symmetry
theorem [17] has been established with a charge-conjugation operator *C*. According to the new
theory [16], if CPT-symmetry is not spontaneously broken, the eigenvalues of the observable
are real.

The aim of the present paper is to further pursue the development of PT-symmetry and to solve the Klein–Gordon (KG) equation of some complex systems. In view of the PT-symmetric formulation, we will apply the Nikiforov–Uvarov (NU) method [18] to solve the KG equation. We have presented exact bound states for a family of exponential potentials, i.e., generalized Hulthén potential which is reducible to the standard Hulthén potential, Woods–Saxon potential and exponential-type screened potential. These have been applied with success to a number of different areas of physical systems. Note that using the quantization of the boundary condition of the states at the origin, Znojil [19] studied another generalized Hulthén and other exponential potentials in non-relativistic and relativistic regions. Dominguez-Adame [20] and Chetouani *et al* [21] also studied relativistic bound states of the standard Hulthén potential. On the other hand, Rao and Kagali [22] investigated the relativistic bound states of the exponential potential by means of the one-dimensional KG equation, and Znojil [23] found the non-relativistic solutions of Schrödinger equation. However, it is well known that for the exponential potential there is no explicit form of the energy expression of bound states for Schrödinger [24], KG [22] and also Dirac [25] equations.

In this paper we will concentrate on the exact solutions of the KG equation for complex exponential-type systems. The structure of the paper is the following. In the next section we introduce a simple algorithm of the KG equation for The PT-symmetric potentials. Here, like the real systems with non-relativistic interactions we can be guided by switching to changes in relativistic interactions. In section 3 we have presented the NU method for exact bound states of generalized Hulthén potential. In section 4 we introduce six different potential cases, three of them are real types and the other three are complex. Since generalized Hulthén potential, the standard Hulthén potential, Woods–Saxon potential and exponential-type screened potential are well defined in nuclear interactions, as an example, we give a simple numerical calculation of the bound states of s-wave pions for real and complex cases. We have also pointed out why the NU method could not be applicable to the exponential-type potential. Finally, conclusions and remarkable facts are discussed in the last section.

2. The Klein–Gordon equation of PT-symmetric potentials

The KG equation for a free particle, in natural units, $\hbar = c = 1$, is [26]

$$\left(\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} + m^2 \right) \psi = 0. \tag{1}$$

Defining $\psi = \theta + \chi$, $i\partial \psi / \partial t = (\theta - \chi)m$ [26], we can write the one-dimensional Schrödinger equation-like KG equation in two components

$$i\frac{\partial \Phi(x)}{\partial t} = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \frac{\wp^2}{2m} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} m + eV(x) \right\} \Phi(x) \tag{2}$$

for a spin-zero particle, in the scalar potential field $V(x)$, where $\wp = p - eA$. Note that, since the kinetic energy term involves the non-Hermitian matrix, $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, the Schrödinger-like Hamiltonian is non-Hermitian and also non-PT-symmetric for any given $V(x)$. It is contrary to the Schrödinger equation for PT-symmetric and Hermitian Hamiltonians. Although the Hamiltonian which is given by equation (2) is non-Hermitian, the transformed one is

$$H' = e^{iS} H e^{-iS} = \eta \sqrt{p^2 + m^2} \tag{3}$$

where η is the diagonal matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $S = -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tanh^{-1}\left(\frac{p^2/2m}{m+p^2/2m} \right)$ with $p = -i\nabla$.

By using the Foldy–Wouthuysen approach [26], for the special case of static external fields, one can find the approximate Schrödinger equation up to the order $1/m^4$ as

$$i\frac{\partial \Phi'}{\partial t} = H'\Phi' \qquad \Phi' = e^{iS}\Phi \tag{4}$$

with

$$H' = \eta \left(m + \frac{\wp^2}{2m} - \frac{\wp^4}{8m^3} + \cdots \right) + eV(x) + \frac{1}{32m^4}[\wp^2, [\wp^2, eV(x)]] + \cdots. \tag{5}$$

In the case of non-relativistic quantum mechanics we have specified minimal coupling of the electromagnetic field, $V(x)$,

$$E \rightarrow i\hbar \frac{\partial}{\partial t} - eV(x). \tag{6}$$

On the other hand, we will be dealing with bound state solutions, i.e., the wave function vanishes at infinity. For any given one-dimensional potential, one can arrive at the time-independent KG equation in the form of Schrödinger equation as [27]

$$\left\{ \frac{d^2}{dx^2} + [(E - V(x))^2 - m^2] \right\} \psi(x) = 0 \tag{7}$$

where $V(x)$ is the one-dimensional vector potential.

Let the potential be written in a complex form,

$$V(x) = V_R(x) + iV_I(x). \tag{8}$$

A potential is called PT-symmetric when

$$PTV(x) = V(x)PT \tag{9}$$

i.e., PT-symmetry condition for a given potential $V(x)$ reads

$$[V(-x)]^* = V(x). \tag{10}$$

We shall study the time-independent KG equation with a family of exponential potentials,

$$V_q(x) = -V_0 \frac{e^{-\alpha x}}{1 - q\, e^{-\alpha x}} \tag{11}$$

which is called generalized Hulthén potential [28]. We have to note that, for some specific q values this potential reduces to the well-known types: such as for $q = 0$, to the exponential potential for $q = 1$ to the standard Hulthén potential and for $q = -1$ to the Woods–Saxon potential. Let us now discuss the limit of very short-ranged potential ($\alpha \rightarrow 0$). In this case the potential is close to the origin

$$V_q(x) \approx \frac{V_0}{q - 1} + \frac{V_0}{(q - 1)^2}\alpha x + O(\alpha^2 x^2) \tag{12}$$

and behaves like a linear potential with a constant shift, $V_0/(q - 1)$, where α denotes the range parameter and V_0 denotes the coupling constant. At this range, one can see the inter-relations between the screened Coulomb potential and generalized Hulthén potential for different parameters; the Hulthén effective potential [29]

$$V_H^{\text{eff}}(r) = -\alpha \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\ell(\ell + 1)\alpha^2}{2} \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \tag{13}$$

which is analytically solvable within the frame of supersymmetric quantum mechanics (SUSYQM) [30] approximates to the screened Coulomb effective potential for small αr as

$$V_{sc}^{\text{eff}}(r) \approx -\frac{e^{-\alpha r}}{r} + \frac{\ell(\ell + 1)}{2r^2}. \tag{14}$$

Working in the framework of SUSYQM, the supersymmetric partner potential corresponding to the screened Coulomb potential $V(r) = -e^{-\alpha r}/r$ may be obtained as [31]

$$V_s(r) = V(r) - \frac{d}{dr}\left(\frac{R_0'(r)}{R_0(r)}\right) \tag{15}$$

in which $R_0(r)$ corresponds to the ground-state wavefunction of the original potential $V(r)$. Moreover, as pointed out by Dutt *et al* [32], the screened Coulomb potential can very well be represented by an effective Hulthén potential.

3. The Nikiforov–Uvarov method

The non-relativistic Schrödinger equation and other Schrödinger-like equations can be solved by using the NU method which is based on the solutions of general second-order linear differential equation with special orthogonal functions [18]. It is well known that for any given one-dimensional or radial potential, the Schrödinger equation can be written as a second-order linear differential equation. However, in the NU method the generalized second-order linear differential equation can be written as

$$\psi''(z) + \frac{\widetilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\widetilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \tag{16}$$

where $\sigma(z)$ and $\widetilde{\sigma}(z)$ are polynomials of degree at most 2, and $\widetilde{\tau}(z)$ is a polynomial of degree at most 1.

Using the transformation

$$\psi(z) = \phi(z)y(z) \tag{17}$$

equation (16) could be reduced to the hypergeometric-type equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{18}$$

whose polynomial solutions are given by the Rodrigues relation

$$y(z, \lambda_n) = y_n(z) = \frac{B_n}{\rho(z)}\frac{d^n}{dz^n}[\sigma^n(z)\rho(z)] \qquad (n = 0, 1, 2, \ldots) \tag{19}$$

where the weight function $\rho(z)$ satisfies the equation

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z). \tag{20}$$

On the other hand, the function $\phi(z)$ satisfies the relation

$$\phi'(z)/\phi(z) = \pi(z)/\sigma(z) \tag{21}$$

with an arbitrary linear polynomial $\pi(z)$ [18].

4. The bound states of generalized Hulthén potential

In order to apply the NU method, we have to write the one-dimensional KG equation as a second-order linear differential equation for the generalized Hulthén potential,

$$\psi_q''(x) + \left[E^2 + 2V_0 E \frac{e^{-\alpha x}}{1 - q\,e^{-\alpha x}} + V_0^2 \frac{e^{-2\alpha x}}{(1 - q\,e^{-\alpha x})^2} - m^2\right]\psi_q(x) = 0 \tag{22}$$

by defining a new variable $z = e^{-\alpha x}$, this equation is reduced to the generalized equation of hypergeometric type which is given by equation (16)

$$\psi_q''(z) + \frac{1 - qz}{z(1 - qz)}\psi_q'(z) + \frac{1}{[z(1 - qz)]^2}[(\gamma^2 - q^2\varepsilon^2 - q\beta^2)z^2$$
$$+ (\beta^2 + 2q\varepsilon^2)z - \varepsilon^2]\psi_q(z) = 0 \tag{23}$$

with $\widetilde{\tau}(z) = 1 - qz$, $\sigma(z) = z(1 - qz)$, $\widetilde{\sigma}(z) = (\gamma^2 - q^2\varepsilon^2 - q\beta^2)z^2 + (\beta^2 + 2q\varepsilon^2)z - \varepsilon^2$. We use the dimensionless abbreviations given by

$$\varepsilon^2 = -\frac{1}{\alpha^2}(E^2 - m^2) \qquad \beta^2 = \frac{2V_0 E}{\alpha^2} \qquad \gamma^2 = \frac{V_0^2}{\alpha^2} \qquad (24)$$

with real $\varepsilon \geqslant 0$ ($E^2 \leqslant m^2$) for bound states.

In the NU method a linear function $\pi(z)$ is defined as

$$\pi(z) = \frac{\sigma'(z) - \widetilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \widetilde{\tau}(z)}{2}\right)^2 - \widetilde{\sigma}(z) + k\sigma(z)} \qquad (25)$$

and in the present case this function becomes

$$\pi(z) = -\frac{qz}{2} \pm \frac{1}{2}\sqrt{[q^2 - 4(\gamma^2 - q^2\varepsilon^2 - q\beta^2) - 4qk]z^2 + 4[k - (\beta^2 + 2q\varepsilon^2)]z + 4\varepsilon^2}. \qquad (26)$$

The constant parameter k can be found from the condition that the expression under the square root has a double zero, i.e., its discriminant is zero. So, there are two possible functions for each k:

$$\pi(z) = -\frac{qz}{2} \pm \begin{cases} \frac{1}{2}[(2q\varepsilon - a)z - 2\varepsilon] & \text{for} \quad k = \beta^2 + a\varepsilon \\ \frac{1}{2}[(2q\varepsilon + a)z - 2\varepsilon] & \text{for} \quad k = \beta^2 - a\varepsilon \end{cases} \qquad (27)$$

where $a = \sqrt{q^2 - 4\gamma^2}$. According to the NU method, with an appropriate choice of the function $\pi(z)$, namely $\pi(z) = \varepsilon - \frac{1}{2}[q + (2q\varepsilon + a)]z$ for $k = \beta^2 - \varepsilon a$, we can define a new function $\tau(z) = \widetilde{\tau}(z) + 2\pi(z)$ which has a negative derivative [18] and is given by

$$\tau(z) = (1 + 2\varepsilon) - [2q + (2q\varepsilon + a)]z. \qquad (28)$$

Then, we have another constant, $\lambda = k + \pi'(z)$, written as

$$\lambda = \beta^2 - \varepsilon a - \frac{1}{2}[q + (2q\varepsilon + a)]. \qquad (29)$$

A new eigenvalue equation for a given λ, $\tau(z)$ and $\sigma(z)$ being defined in [18], is

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \qquad (n = 0, 1, 2, \ldots). \qquad (30)$$

Thus, substituting λ, τ' and σ'' in equation (30), the exact energy eigenvalues of the generalized Hulthén potential are determined as

$$E_n(V_0, q, \alpha) = \frac{V_0}{2q} \pm \kappa\sqrt{\frac{m^2}{4V_0^2 + \kappa^2} - \frac{1}{16q^2}} \qquad (31)$$

where $\kappa = \sqrt{q^2\alpha^2 - 4V_0^2} + q\alpha(2n + 1)$ with $q^2 \geqslant 4V_0^2/\alpha^2$.

Let us now find the corresponding wavefunctions. As stated in equation (17), in the NU method the wavefunction is constructed as a combination of two independent parts. The function $y(z)$ is the polynomial solution of hypergeometric-type equation which is given by equation (18) and described with a weight function [18]. By substituting $\pi(z)$ and $\sigma(z)$ in equation (21) and then solving the first-order differential equation, one can find

$$\phi(z) = z^\varepsilon(1 - qz)^{(a+q)/2q}. \qquad (32)$$

It is easy to find the other part of the wavefunction from the definition of weight function giving

$$\rho(z) = z^{2\varepsilon}(1 - qz)^{a/q} \qquad (33)$$

and substituting in the Rodrigues relation, we obtain

$$y_{nq}(z) = B_{nq} z^{-2\varepsilon}(1 - qz)^{-a/q} \frac{\mathrm{d}^n}{\mathrm{d}z^n}[z^{n+2\varepsilon}(1 - qz)^{n+(a/q)}]. \tag{34}$$

The functions $y_{nq}(z)$ are, up to a numerical factor, the Jacobi polynomials $P_n^{(2\varepsilon,a/q)}(1 - 2qz)$ [33] $(0 \leqslant x \leqslant \infty \to 1 \leqslant z \leqslant 0)$. The wavefunctions $\psi_{nq}(z)$ given by equation (17) are

$$\psi_{nq}(z) = \phi(z)y_{nq}(z) = C_{nq} z^{\varepsilon}(1 - qz)^{(a+q)/2q} P_n^{(2\varepsilon,a/q)}(1 - 2qz). \tag{35}$$

Here C_{nq} is a normalizing constant determined by

$$1 = \int_1^0 |\psi_{nq}(z)|^2 \, \mathrm{d}z = C_{nq}^2 \int_1^0 z^{2\varepsilon}(1 - qz)^{(a+q)/q}\left[P_n^{(2\varepsilon,a/q)}(1 - 2qz)\right]^2 \mathrm{d}z. \tag{36}$$

We now make use of the fact that the Jacobi polynomials can be explicitly written in two different ways $P_n^{(\mu,\nu)}(\xi)$ [33]

$$P_n^{(\mu,\nu)}(\xi) = 2^{-n} \sum_{p=0}^n (-1)^{n-p} \binom{n+\mu}{p}\binom{n+\nu}{n-p}(1 - \xi)^{n-p}(1 + \xi)^p \tag{37a}$$

$$P_n^{(\mu,\nu)}(\xi) = \frac{\Gamma(n+\mu+1)}{n!\Gamma(n+\mu+\nu+1)} \sum_{r=0}^n \binom{n}{r}\frac{\Gamma(n+\mu+\nu+r+1)}{\Gamma(r+\mu+1)}\left(\frac{\xi-1}{2}\right)^r \tag{37b}$$

where $\binom{n}{r}_\Gamma = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Using equations (37), we obtain the explicit expressions for $P_n^{(2\varepsilon,a/q)}(1 - 2qz)$:

$$P_n^{(2\varepsilon,a/q)}(1 - 2qz) = (-1)^n \Gamma(n + 2\varepsilon + 1)\Gamma\left(n + \frac{a}{q} + 1\right)$$

$$\times \sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)!\Gamma\left(\frac{a}{q}+p+1\right)\Gamma(n+2\varepsilon-p+1)} z^{n-p}(1 - qz)^p \tag{38a}$$

$$P_n^{(2\varepsilon,a/q)}(1 - 2qz) = \frac{\Gamma(n + 2\varepsilon + 1)}{\Gamma\left(n + 2\varepsilon + \frac{a}{q} + 1\right)} \sum_{r=0}^n \frac{(-1)^r q^r \Gamma\left(n + 2\varepsilon + \frac{a}{q} + r + 1\right)}{r!(n-r)!\Gamma(2\varepsilon + r + 1)} z^r. \tag{38b}$$

Substituting expression (38a) for one of the Jacobi polynomials in the integral in equation (36) and the expression (38b) for the other and carrying out the integral over z using

$$I_{nq}(p, r) = \int_0^1 z^{n+2\varepsilon+r-p}(1 - qz)^{p+\frac{a}{q}+1} \, \mathrm{d}z \tag{39}$$

one obtains

$$1 = C_{nq}^2 (-1)^{n+1} \frac{\Gamma\left(n + \frac{a}{q} + 1\right)[\Gamma(n + 2\varepsilon + 1)]^2}{\Gamma\left(n + 2\varepsilon + \frac{a}{q} + 1\right)}$$

$$\times \left\{\sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)!\Gamma\left(\frac{a}{q}+p+1\right)\Gamma(n+2\varepsilon-p+1)}\right\}$$

$$\times \left\{\sum_{r=0}^n \frac{(-1)^r q^r \Gamma\left(n + 2\varepsilon + \frac{a}{q} + r + 1\right)}{r!(n-r)!\Gamma(2\varepsilon + r + 1)}\right\} I_{nq}(p, r). \tag{40}$$

The integral given by equation (39) can be evaluated by using the integral representation of the hypergeometric function [34]

$$_2F_1(\alpha_0, \beta_0; \gamma_0; q) = \frac{\Gamma(\gamma_0)}{\Gamma(\alpha_0)\Gamma(\gamma_0 - \alpha_0)} \int_0^1 z^{\alpha_0 - 1}(1 - z)^{\gamma_0 - \alpha_0 - 1}(1 - qz)^{-\beta_0}\,dz$$
$$(\mathrm{Re}(\gamma_0) > \mathrm{Re}(\alpha_0) > 0,\ |\arg(1 - q)| < \pi) \tag{41}$$

which yields

$$B(\alpha_0, 1)\,_2F_1(\alpha_0, \beta_0; \alpha_0 + 1; q) = \int_0^1 z^{\alpha_0 - 1}(1 - qz)^{-\beta_0}\,dz \qquad B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)} \tag{42}$$

when $\gamma_0 = \alpha_0 + 1$. This can easily be verified by expanding $(1 - qz)^{-\beta_0}$ in powers of q and integrating term by term. It is worthwhile here to point out that the hypergeometric function $_2F_1(\alpha_0, \beta_0; \gamma_0; q)$ reduces to

$$_2F_1(\alpha_0, \beta_0; \gamma_0; 1) = \frac{\Gamma(\gamma_0)\Gamma(\gamma_0 - \alpha_0 - \beta_0)}{\Gamma(\gamma_0 - \alpha_0)\Gamma(\gamma_0 - \beta_0)}$$
$$(\mathrm{Re}(\gamma_0 - \alpha_0 - \beta_0) > 0, \mathrm{Re}(\gamma_0) > \mathrm{Re}(\beta_0) > 0) \tag{43}$$

for $q = 1$.

4.1. Real potentials

An inspection of the discrete sequence of real spectra equation given by equation (31) shows that, first considering the real cases, i.e., all parameters (V_0, q, α) are real:

(i) For any given α the spectrum consists of real eigenvalues $E_n(V_0, q, \alpha)$ depending on q. The sign of V_0 does not affect the bound states. It can be seen easily that, while $V_0 \to 0$ in the ground state (i.e. $n = 0$), all energy eigenvalues tend to the value $\approx 0.866m$ for positive q values and $1/\alpha = \lambda_c$, where $\lambda_c = 1/m$ denotes the Compton wavelength of the KG particle. Otherwise, for the same value of α and negative q values, when $V_0 \to 0$, all energy eigenvalues go to zero. If the value of q is increasing, all positive bound states go to zero asymptotically.

(ii) If $4V_0^2/\alpha^2 \leqslant q^2$, there exist bound states, otherwise there are no bound states.

(iii) If $4V_0^2 + \kappa^2 \leqslant 16q^2m^2$, there exist bound states, otherwise there are no bound states. Moreover, this restriction which gives the critical coupling value leads to the result

$$n \leqslant \frac{1}{q\alpha}\left(\sqrt{4q^2m^2 - V_0^2} - \sqrt{\frac{q^2\alpha^2}{4} - V_0^2}\right) - \frac{1}{2} \tag{44}$$

i.e. there are only finitely many eigenvalues. In order that at least one level might exist, it is necessary that the inequality

$$q\alpha + \sqrt{q^2\alpha^2 - 4V_0^2} \leqslant 2\sqrt{4q^2m^2 - V_0^2}$$

is fulfilled. As can be seen from equation (44), there are only two lower-lying states for the KG particle of mass unity when the parameters $\alpha = 1$, $q = \pm 1$ for any given V_0.

(iv) If both conditions (ii) and (iii) are satisfied together, the bound states appear as seen from the energy expression (31). Obviously, for any given positive V_0 value, all possible eigenvalues are in the range $0 \leqslant E_n \leqslant m$ $(-m \leqslant E_n \leqslant 0)$ if the parameter q is positive (negative).

For a more specific case $q = -1$, the generalized Hulthén potential is reduced to the shifted Woods–Saxon potential

$$V(x) = -V_0 + \frac{V_0}{1 + e^{-\alpha x}} \tag{45}$$

and then its energy spectra yield

$$E_n = -\frac{V_0}{2} \pm \left[\sqrt{\alpha^2 - 4V_0^2} - \alpha(2n+1)\right]\sqrt{\frac{m^2}{4V_0^2 + \left[\sqrt{\alpha^2 - 4V_0^2} - \alpha(2n+1)\right]^2} - \frac{1}{16}}. \quad (46)$$

In this case, for any given α, all the eigenstates $E_n \leqslant 0$.

(v) Although the potential reduces to the exponential potential for $q = 0$

$$V(x) = -V_0\, e^{-\alpha x} \quad (47)$$

the eigenvalue expression (31) does not give an explicit form, i.e., the NU method is not applicable to the exponential potential. Note that for this potential there is no explicit form of the energy expression of bound states for Schrödinger [24], KG [22] and also Dirac equations [25].

Now let us discuss why the NU method cannot be applied to the exponential potential. In this case (i.e. $q = 0$), avoiding repetition of the same development, we can write

$$\pi(z) = \pm \begin{cases} (i\gamma z + \varepsilon) & \text{for} \quad k = \beta^2 + 2i\gamma\varepsilon \\ (i\gamma z - \varepsilon) & \text{for} \quad k = \beta^2 - 2i\gamma\varepsilon \end{cases} \quad (48)$$

and the new function $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ yields

$$\tau(z) = (1 + 2\varepsilon) - 2i\gamma z \quad (49)$$

and then, we have another constant, $\lambda = k + \pi'$, given by

$$\lambda = \beta^2 - 2i\gamma\varepsilon - i\gamma. \quad (50)$$

From the last equation we conclude that if and only if $i\gamma$ is real then λ a real constant. Hence, in order to apply the NU method to this potential $\gamma = V_0/\alpha$ should be imaginary. This leads to the result that either V_0 or α must be imaginary. So, we think that this is an open problem nowadays.

4.2. Bound states of s-wave pions

If the depth of the potential is set to $V_0 = Ze^2/4\pi\varepsilon_0 R$, where the nuclear radius is $R = r_0 A^{1/3}$ with $r_0 \approx 1.2$ fm and the number of nucleons $A = N + Z \approx 2.5Z$, we have $V_0 \approx 0.614(e^2/4\pi\varepsilon_0)Z^{2/3}$ (fm^{-1}). In natural units, $\hbar = c = 1$, the unit of charge is defined by choosing $\varepsilon_0 = 1$. The fine structure constant is then $e^2/4\pi\varepsilon_0\hbar c = e^2/4\pi \cong 1/137$. This clearly shows that an electric charge e has no dimension in natural units and is now equal to $\cong 0.303$. In this unit system, we then obtain $V_0 \approx 6.33 \times 10^{-3} m_\pi Z^{2/3}$ since 1 fm $\approx (1/197)$ MeV^{-1} and $m_\pi \cong 139.6$ MeV (i.e., 1 MeV $\cong 7.163 \times 10^{-3} m_\pi$). For example, by means of equation (44), which also gives the critical coupling value, we find

$$\begin{aligned} Z &\leqslant 23 & \text{for} \quad q = 0.10 \\ Z &\leqslant 89 & \text{for} \quad q = 0.25 \\ Z &\leqslant 252 & \text{for} \quad q = 0.50 \end{aligned} \quad (51)$$

i.e., there are no bound state solutions for the s-wave pions (π^- meson) with the Compton wavelength [27] $\alpha = 1/\lambda_\pi = m_\pi$ in the ground state ($n = 0$), because the energy expression which is given by equation (31) becomes imaginary for the real parameters V_0, q, α. When $q = 0.10$, the energy eigenvalue for the s-wave pions at $Z = Z_{\text{critic}} = 22$ is $E_0 \cong 0.946 m_\pi$ since $V_{0\,\text{critic}} \cong 0.05 m_\pi$. The corresponding binding energy is $E_b = E_0 - m_\pi \cong -0.054 m_\pi$. At $Z = 16$, E_0 reaches the maximum value m_π.

4.3. Complex potentials

Now let us consider the cases, namely, at least one of the parameters is imaginary or complex.

(i) If α is a complex parameter, $\alpha = \alpha_R + i\alpha_I$, it may possess real spectra. In this case, for $\alpha_R = 0$, i.e., α is a completely imaginary parameter ($\alpha \to i\alpha$), such potentials are written as a complex function

$$V_q(x) = V_0 \frac{q - \cos(\alpha x) + i \sin(\alpha x)}{q^2 - 2q \cos(\alpha x) + 1} \tag{52}$$

which is PT-symmetric but non-Hermitian. It has real spectra given by

$$E_n = \frac{V_0}{2q} + \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right) \sqrt{\frac{1}{16q^2} - \frac{m^2}{4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right)^2}} \tag{53}$$

if and only if $16q^2 m^2 \leqslant 4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} + q\alpha(2n+1)\right)^2$. The corresponding wavefunctions are

$$\psi_{nq}(z) = \phi(z)y_{nq}(z) = C_{nq} z^{i\varepsilon}(1 - qz)^{(c+q)/2q} P_n^{(2i\varepsilon, c/q)}(1 - 2qz) \qquad (z = e^{-i\alpha x}) \tag{54}$$

where $c = \sqrt{q^2 + 4\gamma^2}$.

The norm of the wavefunction of such a non-Hermitian quantum mechanical system is redefined as [35]

$$\int_0^\infty \psi^*(-x)\psi(x)\,dx = \zeta \qquad (\zeta = \pm 1). \tag{55}$$

$\zeta = 1$ corresponds to the PT-symmetric phase while $\zeta = -1$ corresponds to the PT-antisymmetric phase. So, making the corresponding parameter replacements in equations (39) and (40), i.e. $\varepsilon \to i\varepsilon$ and $a \to c$, we can obtain the normalization constant for the complex PT-symmetric generalized Hulthén potential given by equation (52).

In order to compare the relativistic and non-relativistic binding energies, we need to solve the one-dimensional Schrödinger equation for the generalized Hulthén potential when $\alpha \to i\alpha$. The transformation in this case is $z = e^{-i\alpha x}$ which transforms the one-dimensional Schrödinger equation to the form (in units of $\hbar = m = 1$)

$$\psi_q''(z) + \frac{1 - qz}{z(1 - qz)}\psi_q'(z) + \frac{1}{[z(1 - qz)]^2}\{(qv^2 - q^2\delta^2)z^2 + (2q\delta^2 - v^2)z - \delta^2\}\psi_q(z) = 0 \tag{56}$$

for which

$$\begin{aligned}
&\tilde{\tau}(z) = 1 - qz \qquad \sigma(z) = z(1 - qz) \\
&\tilde{\sigma}(z) = (qv^2 - q^2\delta^2)z^2 + (2q\delta^2 - v^2)z - \delta^2 \\
&\delta^2 = 2E/\alpha^2 \qquad v^2 = 2V_0/\alpha^2.
\end{aligned} \tag{57}$$

Following a procedure analogous to the previous case we see that this time when $\pi(z) = -q(1 + \delta)z + \delta$ is chosen for $k = -q\delta - v^2$,

$$\tau(z) = -q(3 + 2\delta)z + (1 + 2\delta) \tag{58}$$

could be obtained. The energy eigenvalues of the system under consideration are calculated from equation (30) and $\lambda = k + \pi'(z)$ giving

$$E_n = \frac{1}{8q^2\alpha^2}\left[\frac{2V_0 + q\alpha^2(n+1)^2}{(n+1)}\right]^2. \tag{59}$$

Note the positivity of binding energies. Such a paradox has been observed in a few other PT-symmetric potentials [36]. In the present case the wavefunction becomes

$$\psi_{nq}(z) = \phi(z)y_{nq}(z) = C_{nq}z^{\delta}(1 - qz)P_n^{(2\delta,1)}(1 - 2qz) \tag{60}$$

where

$$1 = C_{nq}^2(-1)^{n+1}\frac{(n+1)![\Gamma(n+2\delta+1)]^2}{\Gamma(n+2\delta+2)}\left\{\sum_{p=0}^{n}\frac{(-1)^pq^{n-p}}{p!(p+1)!(n-p)!\Gamma(n+2\delta-p+1)}\right\}$$

$$\times\left\{\sum_{r=0}^{n}\frac{(-1)^rq^r\Gamma(n+2\delta+r+2)}{r!(n-r)!\Gamma(2\delta+r+1)}\right\}\int_0^1 z^{n+2\delta+r-p}(1-qz)^{p+2}\mathrm{d}z \tag{61}$$

$$\int_0^1 z^{n+2\delta+r-p}(1-qz)^{p+2}\,\mathrm{d}z = B(n+2\delta+r-p+1,1)\,{}_2F_1(n+2\delta+r-p+1,$$

$$-p-2; n+2\delta+r-p+2; q). \tag{62}$$

It is seen from table 1 that PT-symmetric but non-Hermitian generalized Hulthén potential both relativistically and non-relativistically generates real and positive bound states. For fixed $V_0 = 0.25$ and chosen α, all the binding energies are decreasing with increasing q. If both parameters q and α are small, i.e. $q < 1$ and $\alpha < 1$, non-relativistic binding energies are higher than the relativistic ones. It is almost notable that there are some crossing points of the relativistic and non-relativistic binding energies for some q values.

(ii) Let V_0 and q be complex parameters, i.e., $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$, where V_{0R}, V_{0I}, q_R and q_I are arbitrary real parameters. Then, one can find different sets of parameters for this condition. In this case, if $V_0 \to iV_0$ and $q \to iq$, then the potential transforms to the form

$$V_q(x) = V_0\frac{[2\cosh^2(\alpha x) - \sinh(2\alpha x) - 1] - i[\cosh(\alpha x) - \sinh(\alpha x)]}{1 + q^2[2\cosh^2(\alpha x) - \sinh(2\alpha x) - 1]} \tag{63}$$

and hence, if and only if $4V_0^2 + \left(\sqrt{q^2\alpha^2 - 4V_0^2} + q\alpha(2n+1)\right)^2 \leqslant 16q^2m^2$ it may possess real spectra as

$$E_n = \frac{V_0}{2q} + \left(\sqrt{q^2\alpha^2 - 4V_0^2} + q\alpha(2n+1)\right)\sqrt{\frac{m^2}{4V_0^2 + \left(\sqrt{q^2\alpha^2 - 4V_0^2} + q\alpha(2n+1)\right)^2} - \frac{1}{16q^2}}. \tag{64}$$

Replacing q and a by iq and ia respectively in equation (35), we obtain the wavefunction for this complex potential as

$$\psi_{nq}(z) = C_{nq}z^{\varepsilon}(1 - iqz)^{(a+q)/2q}P_n^{(2\varepsilon,a/q)}(1 - 2iqz) \tag{65}$$

with $z = e^{-\alpha x}$. From equations (37) and (55), it follows that we need to evaluate integral of the kind

$$I_{nq}(p,r) = \int_0^1 z^{n+2\varepsilon+r-p}(1 - iqz)^{p+\frac{a}{q}+1}\,\mathrm{d}z \tag{66}$$

which then gives [34]

$$I_{nq}(p,r) = B(n+2\varepsilon+r-p+1,1)$$

$$\times {}_2F_1\left(n+2\varepsilon+r-p+1, -p-\frac{a}{q}-1; n+2\varepsilon+r-p+2; iq\right). \tag{67}$$

Table 1. Relativistic and non-relativistic ground state binding energies of a particle of mass unity as a function of q for various values of α in PT-symmetric potential given by equation (52) ($V_0 = 0.25$).

	Klein–Gordon				Schrödinger			
	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 2.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 2.0$
q	$E_0 - m$	$E_0 - m$	$E_0 - m$	$E_0 - m$	E_0	E_0	E_0	E_0
0.097 254	3.837 075	3.055 042	2.641 2305	**2.611 284**	54.156 505	14.532 393	4.714 256	**2.611 284**
0.1	3.748 464	2.976 306	2.568 272	2.542 397	51.257 812	13.781 250	4.500 000	2.531 250
0.265 811	1.611 148	1.149 662	0.945 927	**1.080 831**	7.554 663	2.270 657	1.037 546	**1.080 831**
0.450 844	0.978 413	0.655 912	**0.556 001**	0.767 334	2.744 970	0.923 483	**0.556 001**	0.815 694
0.5	0.882 200	0.584 789	0.503 548	0.726 663	2.257 8125	0.781 250	0.500 000	0.781 250
1.0	0.412 999	0.264 375	0.282 323	0.555 525	0.632 8125	0.281 250	0.281 250	0.632 8125
1.102 178	0.367 528	0.236 548	**0.264 136**	0.541 089	0.532 816	0.247 559	**0.264 136**	0.619 843
1.5	0.250 000	0.168 081	0.219 446	0.504 860	0.313 368	0.170 139	0.222 222	0.586 806
1.941 256	0.178 420	**0.128 811**	0.193 348	0.482 984	0.204 883	**0.128 811**	0.197 684	0.566 464
2.0	0.171 497	0.125 097	0.190 836	0.480 840	0.195 3125	0.125 000	0.195 3125	0.564 453
2.5	0.127 325	0.101 629	0.174 668	0.466 858	0.137 8125	0.101 250	0.180 000	0.551 250
4.455 606	0.060 152	**0.065 601**	0.148 193	0.443 102	0.061 053	**0.065 601**	0.154 629	0.528 448
5.0	0.052 311	0.061 169	0.144 706	0.439 876	0.052 8125	0.061 250	0.151 250	0.525 3125
7.5	0.033 352	0.049 855	0.135 445	0.431 175	0.033 368	0.050 139	0.142 222	0.516 806
10.0	0.025 336	0.044 636	0.130 953	0.426 879	0.025 3125	0.045 000	0.137 8125	0.512 578

It is important that this kind of potentials are non-PT-symmetric and also non-Hermitian but have exact real spectra. As seen from table 2, all eigenvalues are negative, i.e., restricted to those within the well depth. Also, there is a restriction between the potential parameters for relativistic binding energies. Again, there are some crossing points of the relativistic and non-relativistic binding energies for some q values.

Moreover, it is worthwhile to point out here that, as could be seen from equation (44) even for this complex potential taking the parametric values given in section 4.2, again equation (51) could be obtained giving the critical Z values in the ground state.

(iii) It is interesting to note that when all three parameters V_0, q and α are imaginary at the same time, we obtain the potential as

$$V_q(x) = V_0 \frac{q - \sin(\alpha x) - \mathrm{i}\cos(\alpha x)}{q^2 - 2q\sin(\alpha x) + 1}. \tag{68}$$

This form of the potential has a $\pi/2$ phase difference with respect to the potential in (i), it is also non-PT-symmetric and non-Hermitian but has real spectra

$$E_n = \frac{V_0}{2q} - \left(\sqrt{q^2\alpha^2 + 4V_0^2} - q\alpha(2n+1)\right) \sqrt{\frac{1}{16q^2} - \frac{m^2}{4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} - q\alpha(2n+1)\right)^2}} \tag{69}$$

if and only if $16q^2m^2 \leqslant 4V_0^2 - \left(\sqrt{q^2\alpha^2 + 4V_0^2} - q\alpha(2n+1)\right)^2$. Now again referring back to equation (35), the corresponding wavefunctions $\psi_{nq}(z)$ are identified in the form

$$\psi_{nq}(z) = C_{nq} z^{\mathrm{i}\varepsilon}(1 - \mathrm{i}qz)^{(c+q)/2q} P_n^{(2\mathrm{i}\varepsilon, c/q)}(1 - 2\mathrm{i}qz) \tag{70}$$

with $c = \sqrt{q^2 + 4\gamma^2}$ and $z = \mathrm{e}^{-\mathrm{i}\alpha x}$. As before, the integral which appears in normalization can be calculated as [34]

$$\mathrm{I}_{nq}(p, r) = \int_0^1 z^{n+r-p+2\mathrm{i}\varepsilon}(1 - \mathrm{i}qz)^{p+\frac{c}{q}+1}\,\mathrm{d}z$$

$$= B(n+r-p+1+2\mathrm{i}\varepsilon, 1)_2F_1\left(n+r-p+1+2\mathrm{i}\varepsilon,\right.$$

$$\left. - p - \frac{c}{q} - 1; n+r-p+2+2\mathrm{i}\varepsilon; \mathrm{i}q\right) \tag{71}$$

with $\mathrm{Re}(n+r-p+1+2\mathrm{i}\varepsilon) > 0$, i.e., $n > p - r - 1$ and $|\arg(1 - \mathrm{i}q)| < \pi$.

5. Results and discussion

We have solved the Klein–Gordon equation for the generalized Hulthén potential in relativistic quantum mechanics. According to the second condition of complex quantum mechanics [13], the eigenfunctions of the cases obtained by $\alpha \to \mathrm{i}\alpha$ are simultaneously eigenstates of the PT operator. As should be expected (see equation (31)), for any given set of potential parameters α and V_0, although the energy levels of Woods–Saxon potential ($q = -1$) are negative, the energy levels of standard Hulthén potential ($q = 1$) are positive. On the other hand, we can state that our results are not only interesting for theoretical physicists but also for experimental physicists, because the results are exact and more general. As an example, an application to the pionic systems is given.

We have already mentioned that we have found some simple relations among the potential parameters for bound states. We show that it is possible to obtain relativistic bound states

Table 2. Relativistic and non-relativistic ground state binding energies of a particle of mass unity as a function of q for various values of α in non-PT-symmetric potential given by equation (63) ($V_0 = 0.25$).

	Klein–Gordon				Schrödinger			
	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 2.0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.0$	$\alpha = 2.0$
q	$E_0 - m$	$E_0 - m$	$E_0 - m$	$E_0 - m$	E_0	E_0	E_0	E_0
0.1	–	–	–	–	−48.757 812	−11.281 250	−2.000 000	−0.031 250
0.301 777	–	–	–	**−0.171 573**	−5.083 920	−0.989 617	−0.053 932	**−0.171 573**
0.5	–	–	−0.088 562	−0.500 000	−1.757 8125	−0.281 250	0.000 000	−0.281 250
0.580 679	–	–	**−0.002 413**	−0.569 469	−1.275 399	−0.186 698	**−0.002 413**	−0.307 904
1.0	–	−0.179 0295	−0.029 196	−0.750 000	−0.382 8125	−0.031 250	−0.031 250	−0.382 8125
1.135 857	–	−0.051 402	**−0.039 172**	−0.779 902	−0.285 309	−0.018 087	**−0.039 172**	−0.396 006
1.5	–	−0.008 327	−0.059 055	−0.833 333	−0.146 701	−0.003 472	−0.055 555	−0.420 139
2.0	−0.233 161	−0.000 160	−0.076 107	−0.875 000	−0.070 3125	0.000 000	−0.0703 125	−0.439 453
2.066 561	−0.152 267	**−0.000 032**	−0.077 819	−0.879 026	−0.064 403	**−0.000 032**	−0.071 830	−0.441 342
2.5	−0.061 180	−0.000 860	−0.086 910	−0.899 999	−0.037 8125	−0.001 250	−0.080 000	−0.451 250
4.489 155	−0.005 923	**−0.009 608**	−0.107 029	−0.944 310	−0.004 778	**−0.009 608**	−0.098 706	−0.472 543
5.0	−0.003 449	−0.011 333	−0.109 699	−0.950 000	−0.002 8125	−0.011 250	−0.101 250	−0.4753 125
7.5	−0.000 058	−0.017 107	−0.117 629	−0.966 666	−0.000 035	−0.016 806	−0.108 889	−0.483 472
10.0	−0.000 288	−0.020 387	−0.121 655	−0.974 999	−0.000 3125	−0.020 000	−0.1128 125	−0.487 578

of complex quantum mechanical formulation. Thus the relativistic bound state spectra of the generalized Hulthén potential exhibit the effects of a well of depth V_0, potential range parameter α and shape parameter q. The remarkable fact is that, for fixed potential well depth and range parameter α, there are the critical shape parameters q, for which both energies are at the same levels. Nevertheless, for small α and large q, the relativistic binding energies are close to the non-relativistic energy. We show that it is possible to obtain relativistic representation of the PT-symmetric quantum mechanical formulation.

Acknowledgments

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